Chapter 5.2 Vertex Coloring

Theorem [Brook's]. Let G be a connected graph, that is not a complete graph or an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Notice that Brook's theorem tells us that $\chi(G) \leq \Delta(G) + 1$ holds with equality iff G is a complete graph or an odd cycle.

Let's prove Brook's theorem. Let G be a connected graph that is not complete or an odd cycle. Also assume that we have proved the theorem for all smaller graphs (we are proving by induction on the number of vertices). Let $\Delta(G) = \Delta$.

1: Show that the Brook's theorem holds if $\Delta = 2$

Solution: If $\Delta = 2$, then G is a cycle or a path and checking the Brook's claim is straightforward.

So we assume $\Delta \geq 3$.

2: Show that G is 2-connected. (use induction)

Solution: If G not 2-connected, one can color all blocks of G separately with Δ colors and combine them by permuting colors. Recall that block is a maximal 2-connected subgraph.

3: Prove the case where G has a vertex v of degree less than Δ (greedy coloring with v last)
Hint: Use a spanning tree to make an ordering where everyone but v has a neighbor that comes later.

Solution: Take any spanning tree T of G. Orient all edges of T towards v. Now order the vertices of G such that if xy is an edge of T oriented in this direction, then x is in the ordering before y. Hence v is last in the ordering and every vertex has at least one neighbor behind in the ordering. Now try to color vertices according to this ordering. Every vertex has to avoid color of at most $\Delta - 1$ other vertices, that are in the ordering before. So Δ colors is enough.

Now we assume G is Δ-regular. We still want to use greedy coloring, but guarantee that the last vertex has 2 neighbors with the same color.

4: Assume that there is a vertex v such that $G - v$ is 2-connected. Prove Brook's Theorem.
Hint: Start as: Take v and any vertex y in distance 2 from it (why it exists?). They have a common neighbor z. Since $G - v$ is 2-connected, $G - v - y$ is connected. Hence there is an orientation of edges of $G - v - y$ that all go towards z.

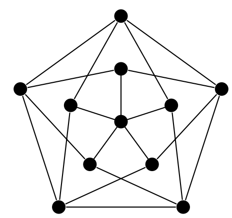
Solution: So we there is an ordering of $G - v - y$ where z is last and all vertices but z have a neighbor behind in the ordering. Now we create the final ordering by putting the vertices v and y first. Notice that in a coloring, v and y get the same color since they come first and they are not adjacent. So z has Δ neighbors that are already colored when coloring z, but two of them have the same color. Hence Δ colors is enough.

5: Assume that there is a vertex v such that $G - v$ is not 2-connected. Prove Brook's Theorem. Notice that $G - v$ is still connected. Consider block decomposition of $G - v$ and see where are neighbors of v.

Solution: Since G is 2-connected, v has to have neighbors in the *end*-blocks. Take neighbors x and y of v in two different end-blocks. Notice that $G - x - y$ is still connected. Create an ordering where x and y are first and v is last.

Question: Do you need a large clique in a graph G for a large chromatic number?

6: What is the chromatic number of the Grötzsch's graph? Notice it is triangle-free.



Solution: It is 4. There is a coloring with 4 colors. For showing that 3 is not enough, try to do a 3-coloring. First color the outer 5-cycle. Then vertices in the inner 5-cycle must still contain all three colors. This kills the last color for the middle vertex.

Theorem (Erdős) For every k, ℓ there exists a graph of girth ℓ and chromatic number k.

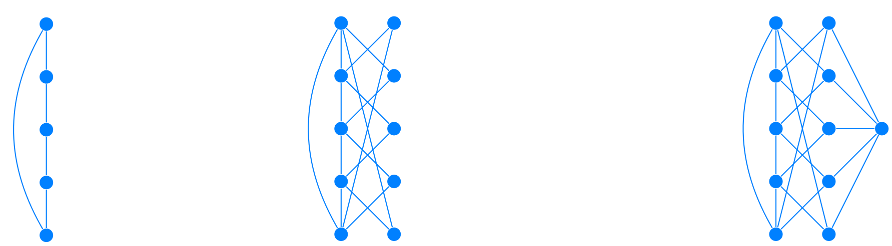
The proof is probabilistic and we skip it for now.

Mycielski construction is a construction to create a triangle free graph of an arbitrary chromatic number.

Start with a graph G, duplicate every vertex and connect new vertex to the duplicates.

7: Apply the Mycielski operation on C_5.

Solution:



C_5 C_5 duplicated vertrices   The whole thing

Notice that this graph is isomorphic to the Grötzsch's graph.

8: Let G be a graph with $\chi(G) = k$ and c be a k-coloring of G. Show that for every color z, there exists a vertex $v \in V(G)$ colored z such that all remaining $k - 1$ colors are on neighbors of v.
Hint: Show that if the conclusion is not true, then $\chi(G) < k$.

Solution: Suppose for contradiction that there exists a color z such that for every vertex v of color z exists a color $a_v \neq z$ that is not on any of the neighbors of v. The v can be recolored from z to a_v. If we do it for all vertices colored by z, there will be none left. Notice that vertices colored by z form an independent set so this operation will not create new conflicts.

9: Show that the Mycielski construction is increasing the chromatic number.

Solution: Suppose the original graph G that went into the construction has $\chi(G) = k$. If we k-color it, for every color there is a vertex c such that its neighbors have all remaining $k-1$ colors. Hence the duplicate of v must have the color of v. The last vertex added will have neighbors of all k colors. Hence the graph that came from the construction is not k-colorable and at least one extra color is needed.

Now we try to *construct* all graphs with chromatic number at least k.

k-constructible graphs are obtained recursively as follows:

1. K_k is k-constructible
2. If G is k-constructible and two vertices x, y of G are non-adjacent, then also $(G+xy)/xy$ is k-constructible.
3. If G_1,G_2 are k-constructible and there are vertices x, y_1, y_2 such that $G_1 \cap G_2 = \{x\}$ and $xy_1 \in E(G_1)$ and $xy_2 \in E(G_2)$, then also $(G_1 \cup G_2) - xy_1 - xy_2 + y/-1y_2$ is k-constructible.

10: Show that all odd cycles are 3-constructible.

Solution: Start with triangles. Next one can use 3 to merge two triangles together and get a 5-cycle. Notice that merging with triangle always adds 2 vertices, so we keep getting odd cycles.

11: Show that all k-constructible graphs are at least k-chromatic. That is, they cannot be colored by $k-1$ colors.

Solution: K_k clearly needs k colors

For 2, if there is an ℓ coloring c of $(G+xy)/xy$ then one can find an ℓ-coloring of G by simply reusing the color $c(v_{xy})$ on x and y. Here v_{xy} is the vertex obtained by identification of x and y.

For 3, any ℓ-coloring c of the resulting graph gives $c(y_1) \neq c(y_2)$. Hence one of them is not the same as $c(x)$. This means either G_1 or G_2 have ℓ-coloring.

Hence $\ell \geq k$.

Theorem (Hajós 1961)
Let G be a graph and $k \in \mathbb{N}$. Then $\chi(G) \geq k$ if and only if G has a k-constructible subgraph.

Notice that adding things to G will not decrease the chromatic number. So we cannot give an exact description of G but we show it has a subgraph that makes the chromatic number large.

Proof Let G be a graph with $\chi(G) \geq k$. We want to show it has a k-constructible subgraph. Suppose for contradiction G has no k-constructible subgraph. Take G minimal with respect to the number of vertices. Subject to that, take G maximal with respect to the number of edges (adding any edge causes existence of a k-constructible subgraph).

We would like to find a triple of vertices inducing one edge and then use 3. for finding some k-constructible subgraph.

12: Show that G is not a complete r-partite graph.

Solution: If G is complete r partite then G is r-colorable. If $r < k$, it is a contradiction with $\chi(G) \geq k$. If $r \geq k$, then G contains K_k as a complete subrgaph. This is a contradiction since now G contains a k-constructible subgraph K_k.

13: Show that G contains x, y_1, and y_2 such that $y_1y_2 \in E(G)$ but $xy_1, xy_2 \notin E(G)$.

Solution: Since G is not a complete r-partite graph, relations of a non-edge is not an equivalence. There must be something that shows it is not an equivalence class and that something is exactly x, y_1, and y_2.

By the maximality of G, $G + xy_i$ contains a k-constructible subgraph H_i.

14: Finish the proof by finding a k-constructible subgraph in G. Start by applying 3. on H_1 and H_2. What if H_1 and H_2 have common vertices?

Solution: Take copies of H_1 and H_2. These are k-constructible. Applying 3. also creates a k-constructible subgraph. Now x is not adjacent to y_1 or y_2. For every vertex shared by H_1 and H_2, apply operation 2 to identify them. The result is a subgraph of G.

15: [Open problem] Reed's conjecture: $\chi(G) \leq \lceil(\omega(G) + \Delta(G) + 1)/2\rceil$